REFERENCES

1. MOISEYEV N.N., On the non-uniqueness of the possible forms of steady flows of a heavy fluid for Froude numbers close to unity. PMM, 21, 6, 1957.
2. GUZEVSKII L.G., The flow of a heavy fluid of finite depth past obstacles, in: Dynamics of a Continuous Medium with Interfaces, Izd Chuvash. Univ., Cheboksary, 1982.
3. KOTLYAR L.M. and TROYEPOL'SKAYA O.V., Flow of a jet of heavy fluid through an opening in a horizontal wall, in the presence of a free surface. Proceedings of the Seminar on Boundary Value Problems, Izd. Kazan. Univ. 22, 1985.
4. KOCHIN N.E., KIBEL' I.A. and ROZE N.V., Theoretical Hydromechanics, 1, Fizmatgiz, Moscow, 1963.
5. IVANOV A.N., Hydrodynamics of Developed Cavitational Flows. Sudostroyeniye. Leningrad, 1980.
6. AMROMIN E.L., BUSHKOVSKII V.A. and DIANOV D.I., Developed cavitation behind a disc in a vertical pipe. Izv. Akad. Nauk SSSR, MZhG, 5, 1983.
7. ZABOREIKO P.P., KOSHELEV A.I., KRASNOSEL'SKII M.A., et al., Integral Equations. Nauka, Moscow, 1968.
8. TERENT'YEV A.G. and AFANAS'YEV K.E., Numerical Methods in Hydromechanics. Izd. Chuvash. Univ. Cheboksary, 1987.

Translated by L.K.

PMM U.S.S.R., Vol.54, No.1, pp. 137-142, 1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

LOWER ESTIMATES OF THE CHARACTERISTIC FREQUENCIES OF THE OSCILLATIONS OF A LIQUID WITH A FREE SURFACE IN CHANNELS OF ARBITRARY CROSS-SECTION*

V.I. TARAKANOV

Lower estimates are obtained for the leading characeristic frequency of oscillations of a liquid in a channel of arbitrary cross-section with several sections of the free surface of the liquid. The case of oscillations in the plane of the cross-section of the channel is considered. The domain occupied by the cross-section can be multiply connected and bounded by a piecewise smooth curve. The derivation of the estimates is not connected with the need to find standard domains and is not based on variational methods /1-3/.

1. The boundary eigenvalue problem

$$\Delta U \equiv U_{,xx} + U_{,yy} = 0, \quad x, y \in D \tag{1.1}$$

$$\frac{\partial u}{\partial n} = \omega^2 U, \quad x, y \in \Gamma_\alpha; \quad \frac{\partial U}{\partial n} = 0, \quad x, y \in \Gamma/\Gamma_\alpha \tag{1.2}$$

is considered for a multiply connected domain $D \subset R^2(x, y)$ bounded by a piecewise-smooth curve Γ consisting of a number of closed curves. Γ has m segments Γ_j for $j = 1, 2\cdots m$, where boundary conditions corresponding to the conditions on the free surface of the liquid are given.

$$\Gamma_j: y = h_j, \quad a_j < x < b_j, \quad j = 1, 2\cdots m, \quad \Gamma_\alpha = \Gamma_1 \cup \Gamma_2 \cup \cdots \Gamma_m$$

It is assumed that the segments of the free surface can be placed at different levels $y = h_j$ (for example, to maintain different pressures of gases over different sections of the surface). In the general case, the segments Γ_j may belong to different closed curves of the contour Γ. Apart from a dimensional factor, ω is identical with the characteristic frequency of oscillations of the liquid. Thus, in what follows it is called simply the frequency of characteristic oscillations of the liquid.

---

*Prikl. Matem. Mekhan., 54, 1, 165-170, 1990

The eigenfunctions of problem (1.1) are sought in the class

$$U \in C^{\infty}(D) \cap C(D \cup \Gamma), \quad E \equiv \int_D (U_{,x}^2 + U_{,y}^2)\, d\sigma \equiv \int_\Gamma U U_{,n}\, ds < \infty \tag{1.3}$$

2. As a preliminary, we will consider the special case of a star-shaped domain D. By a star-shaped domain with respect to a point $Q(\xi, \eta) \in D$ we mean a domain whose boundary is intersected only once by any ray starting from Q in such a way that the ray is not tangent to the boundary. Apart from Cartesian coordinates, a system of polar coordinates ρ, φ with centre at Q is introduced: $x - \xi = \rho\cos\varphi$ and $y - \eta = \rho\sin\varphi$. The segment of Γ contained between the rays $\varphi = \beta$ and $\varphi = \beta + \alpha$ is denoted by $\Gamma(\alpha, \beta)$. The domain D is characterized by the following parameters expressed in terms of the distance between two points $M(x, y)$ and $Q(\xi, \eta)$:

$$r = \min_{x, y \in \Gamma} \rho, \quad R = \max_{x, y \in \Gamma(\alpha, \beta)} \rho, \quad \rho = \sqrt{(x-\xi)^2 + (y-\eta)^2} \tag{2.1}$$

$$c = \frac{1}{2}\frac{R}{r}\left(1 - \frac{r^2}{R^2}\right), \quad b = c\frac{R}{r}, \quad a = r \min_{x, y \in \Gamma(\alpha, \beta)} \rho^{-1}\cos n\rho$$

where $\cos n\rho$ is the cosine function of the angle between the normal to Γ and the extension of the ray connecting Q and M. Since the domain is star-shaped, the condition $a > 0$ holds.

Let an arbitrary function

$$\Phi(x, y) = U(x, y) - U(Q) \tag{2.2}$$

be given in D, where U is a function belonging to the class defined by (1.1) and (1.3).

Theorem 1. For any function Φ from the class defined by (1.1), (1.3), and (2.2), the estimate

$$\int_{\Gamma(\alpha, \beta)} \Phi^2\, ds \leqslant \varkappa E, \quad E \equiv \int_D (\Phi_{,x}^2 + \Phi_{,y}^2)\, d\sigma \tag{2.3}$$

$$\varkappa \equiv \varkappa(\alpha, \beta, Q) \equiv \frac{r}{a}\max\left\{3,\ 34\frac{c+2b}{c+b}\sqrt{\frac{\alpha}{2\pi}};\ 4(b+c)\right\} \tag{2.4}$$

holds on the boundary.

Proof. In D we choose a circle δ of radius r, centre Q, and boundary γ. The derivatives $|\Phi_{,x}|$ and $|\Phi_{,y}|$ satisfy the Courant inequality /4/

$$|\Phi_{,x}|, |\Phi_{,y}| \leqslant d^{-1}\sqrt{E_0/\pi}, \quad x, y \in \delta, \quad \xi, \eta \in \gamma \tag{2.5}$$

$$E_0 \equiv \int_\delta (\Phi_{,x}^2 + \Phi_{,y}^2)\, d\sigma \leqslant E$$

where $d = d(x, y, \xi, \eta)$ is the minimum distance between $M(x, y)$ and γ.

Since inequality (2.5) is independent of rotations of the coordinate axes, the estimate also holds in polar coordinates:

$$|\Phi_{,\rho}| \leqslant (r-\rho)^{-1}\sqrt{E_0/\pi}, \quad E_0 = \int_\delta (\Phi_{,\rho}^2 + \rho^{-2}\Phi_{,\varphi}^2)\, d\sigma \tag{2.6}$$

The estimate

$$|\Phi|^2 \equiv \left|\int_0^\rho \Phi_{,\rho}\, d\rho\right|^2 \leqslant \frac{E_0}{\pi}\ln^2\left(1 - \frac{\rho}{r}\right), \quad 0 < \rho < r \tag{2.7}$$

follows from (2.6).

For a sector $\sigma \subset \delta$ σ of a circle such that $\varepsilon < \rho < r$ and $\beta < \varphi < \beta + \alpha$, the Green integral identity

$$\int_\lambda \Psi_{,n}\Phi^2\, ds = \int_\sigma [\Delta\Psi\Phi^2 + 2\Phi(\Psi_{,x}\Phi_{,x} + \Psi_{,y}\Phi_{,y})]\, d\sigma$$

can be written down, where λ is the boundary of the sector, and Ψ is an arbitrary function with continuous second-order derivatives. If Ψ depends only on ρ, then the relation takes the form

$$\int\limits_{\gamma(\alpha,\beta)} \Psi_{,n}\Phi^2\,ds + \int\limits_{\gamma(\alpha,\beta,\varepsilon)} \Psi_{,n}\Phi^2\,ds = \int\limits_{\delta} [\Delta\Psi\Phi^2 + 2\Phi\Phi_{,\rho}\Psi_{,\rho}]\,d\sigma \tag{2.8}$$

$$\gamma(\alpha,\beta) \subset \lambda,\ \gamma(\alpha,\beta,\varepsilon) \subset \lambda;\ \gamma(\alpha,\beta){:}\ \rho = r,\ \beta < \varphi < \beta + \alpha$$
$$\gamma(\alpha,\beta,\varepsilon){:}\ \rho = \varepsilon,\ \beta < \varphi < \beta + \alpha$$

Setting $\Psi = \ln\rho$ in (2.8), using (2.7) and (2.6), and passing to the limit as $\varepsilon \to 0$, we obtain the estimate

$$\frac{1}{r}\int\limits_{\gamma(\alpha,\beta)} \Phi^2\,ds \leqslant 2\sqrt{E_0\int\limits_{\delta}\frac{1}{\rho^2}\Phi^2\,d\sigma} \leqslant \tag{2.9}$$
$$2\sqrt{2}E_1\left[\int\limits_0^1 x^{-1}\ln^2(1-x)\,dx\right]^{1/2} = 3.34E_1, \quad E_1 = E_0\sqrt{\frac{\alpha}{2\pi}}$$

To derive (2.9), the Hölder inequality is used and the integral on the right-hand side is replaced by its numerical value.

Next, we write down the integral identity (2.8) for the domain $G \subset D$ bounded by the curves $\Gamma(\alpha,\beta)$ and $\gamma(\alpha,\beta)$ and by the rays $\varphi = \beta$ and $\varphi = \beta + \alpha$:

$$\int\limits_{\Gamma(\alpha,\beta)} \Psi_{,n}\Phi^2\,ds + \int\limits_{\gamma(\alpha,\beta)} \Psi_{,n}\Phi^2\,ds = \int\limits_{G} [\Delta\Psi\Phi^2 + 2\Phi\Phi_{,\rho}\Psi_{,\rho}]\,d\sigma \tag{2.10}$$

Substituting $\frac{1}{4}\rho^2 - \frac{1}{2}r^2\ln(\rho/r)$ for Ψ in this identity and using the notation given in (2.1), we find that

$$I \leqslant br^2 \int\limits_{\Gamma(\alpha,\beta)} \Phi^2\frac{\cos n\rho}{\rho}\,ds + 2cr(E_2 I)^{1/2} \tag{2.11}$$

$$I = \int\limits_G \Phi^2\,d\sigma, \quad E_2 = \int\limits_G (\Phi_{,\rho}^2 + \rho^{-2}\Phi_{,\varphi}^2)\,d\sigma, \quad E_0 + E_2 \leqslant E \tag{2.12}$$

If we now substitute $\ln\rho$ for Ψ in (2.10), we shall find using (2.9) that

$$\int\limits_{\Gamma(\alpha,\beta)} \Phi^2\frac{\cos n\rho}{\rho}\,ds \leqslant 3{,}34E_1 + \frac{2}{r}(E_2 I)^{1/2} \tag{2.13}$$

The inequality

$$I \leqslant 3.34br^2E_1 + 2(b+c)\,r\,(IE_2)^{1/2}$$

can be derived from the system of inequalities (2.11) and (2.13). Hence, solving the quadratic equation, we finally find that

$$I \leqslant (b+c)\,r\sqrt{E_2} + [r^2(b+c)^2E_2 + 3.34br^2E_1]^{1/2} \tag{2.14}$$

It follows from (2.13) and (2.14), taking (2.1) into account, that

$$\frac{a}{r}\int\limits_{\Gamma(\alpha,\beta)} \Phi^2\,ds \leqslant 3{,}34E_1 + 2(b+c)E_2 + 2[(b+c)^2E_2^2 + 3.34bE_1E_2]^{1/2} \leqslant$$
$$2[(b+c)^2E_2^2 + 3.34bE_1E_2 + (3.34)^2 8^{-1}b^2(b+c)^{-2}E_1^2]^{1/2} + 3.34E_1 +$$
$$2(b+c)E_2 = 3.34(c+2b)(b+c)^{-1}E_1 + 4(b+c)\,E_2$$

Now, using inequality (2.12), we obtain the result of the theorem.

On the basis of (2.4), the parameter $\varkappa$ in estimate (2.3) for any segment of the boundary of an arbitrary star-shaped domain is easy to evaluate by a geometrical argument. Thus, for a square with sides l, a sector of the form of a quarter of a circle or radius l, an equilateral triangle with slides l, and for a right-angled isosceles triangle with hypotenuse l, the parameters $\varkappa$ have the following values

$$\varkappa = 3.4l,\ \varkappa = 4.0l,\ \varkappa = 10.3l,\ \varkappa = 22.8l \tag{2.15}$$

provided each of the points Q_i is chosen to be at the centre of the corresponding inscribed circle and the segment $\Gamma(\alpha,\beta)$ is chosen to be either a side of the square, or a radial side of the sector, or a side of the equilateral triangle, or the hypotenuse of the right-angled triangle, respectively.

3. Using Theorem 1, we can obtain a more general result for an arbitrary domain D, which is not necessarily star-shaped and has boundary Γ. We denote by Γ_α an arbitrary part of Γ: $\Gamma_\alpha \subset \Gamma$. It is assumed that there are n arbitrary points $Q_i(\xi_i, \eta_i) \subset D$ for $i = 1, 2 \cdots n$ and n star-shaped domains $D_i \subset D$ with respect to the points Q_i such that boundaries Γ_i of the domains have common points with Γ and the relations

$$\Gamma_\alpha = \Gamma_{1\alpha} \cup \Gamma_{2\alpha} \cup \ldots \Gamma_{n\alpha}, \quad \Gamma_{i\alpha} = \Gamma_i \cap \Gamma_\alpha, \quad i = 1, 2 \ldots n \tag{3.1}$$

are satisfied.

An arbitrary central point Q_0 is chosen in D so that it can be connected with each of the points Q_i for $i = 1, 2 \cdots n$ by a curve $\gamma_i \in D$ contained in D that does not touch the boundary Γ.

Three parameters $\varkappa_i$, l_i, and q_i are associated with each domain D_i where $\varkappa_i$ is defined by (2.4) for any star-shaped domain D_i with $\Gamma_{i\alpha}$ used as the segment of the boundary of D_i that appears in the definition of $\varkappa$, and where l_i and q_i are given by the relations

$$l_i = \int_{\Gamma_{i\alpha}} ds, \quad q_i = \int_{\gamma_i} \rho^{-1}(s)\, ds \tag{3.2}$$

Here ds is an element of the arc of the curve, and $\rho(s)$ is the minimum distance between $x(s), y(s) \in \gamma_i$ and the points of Γ. The function $\Phi_i = U(x, y) - U(Q_i)$ is introduced for each domain, where U is a function belonging to the class described by Eqs.(1.1) and (1.3). The integral

$$E_i = \int_{D_i} (\Phi^2_{i,x} + \Phi^2_{i,y})\, d\sigma \equiv \int_{D_i} (U^2_{,x} + U^2_{,y})\, d\sigma \tag{3.3}$$

$$\sum_{i=1}^{n} E_i \leqslant E = \int_D (U^2_{,x} + U^2_{,y})\, d\sigma$$

is denoted by E_i.

Theorem 2. For the function $\Phi = U(x, y) - U(Q_0)$ belonging to the class described by (1.1) and (1.3), the estimate

$$\int_{\Gamma_\alpha} \Phi^2\, ds \leqslant \varkappa E \tag{3.4}$$

$$\varkappa = \max_i \varkappa_i + 2\sqrt{n/\pi}\, \max_i q_i \sqrt{l_i \varkappa_i} + \pi^{-1} \sum_1^n l_i q_i^2 \tag{3.5}$$

holds.

Proof. Using Courant's inequality (2.5) for D and taking into account that the inequality is independent of the rotation of the coordinate axes, one can write the following:

$$|\Phi_{,s}| \leqslant \rho^{-1}(s)\sqrt{E/\pi}, \quad x(s), y(s) \in \gamma_i$$

Hence, using notation (3.2), we have

$$|\Phi(Q_i)| = \left|\int_{\gamma_i} \Phi_{,s}\, ds\right| \leqslant \int_{\gamma_i} |\Phi_{,s}|\, ds \leqslant q_i \sqrt{E/\pi} \tag{3.6}$$

Taking into account that inequalities (2.3) hold for $\Phi_i \equiv \Phi(x, y) - \Phi(Q_i)$, we find that

$$\int_{\Gamma_{i\alpha}} \Phi_i^2\, ds \leqslant \varkappa_i E_i \tag{3.7}$$

On the basis of relations (3.6) and (3.7) we obtain the estimate

Using the Hölder inequalities for the sum $\sqrt{E_1}+\cdots+\sqrt{E_n}$, we obtain the theorem.

In the special case when $\Gamma_\alpha = \Gamma$, the result of the theorem with a different expression for the coefficient was proved in /5/ for a domain with a smooth boundary. In practice, one can obtain estimate (3.4), (3.5) for an arbitrary domain using a collection of standard star-shaped domains, for which $\varkappa$ is found in advance from the relations stated in Theorem 1.

4. On the basis of Theorem 2, using the notation of this theorem, we obtain a lower estimate for the leading characteristic frequency of oscillations of the liquid.

Theorem 3. The leading characteristic frequency ω_1 of oscillations of the liquid, which can be determined from the solution of problem (1.1), (1.2), is bounded from below by

$$\omega_{10} = \varkappa^{-1/2}, \quad \omega_1 \geqslant \omega_{10}$$

where $\varkappa$ is defined by (3.5), and the segment Γ_α of the boundary represents the free surface of the liquid.

Proof. It follows from (3.4) and from the boundary conditions of the problem that

$$\int\limits_{\Gamma_\alpha} \Phi^2\, ds \leqslant \varkappa \int\limits_{\Gamma} \Phi\Phi_{,n}\, ds = \varkappa \int\limits_{\Gamma_\alpha} \Phi\Phi_{,n}\, ds \leqslant \varkappa \sqrt{\int\limits_{\Gamma_\alpha} \Phi^2\, ds} \sqrt{\int\limits_{\Gamma_\alpha} \Phi_{,n}^2\, ds} \tag{4.1}$$

$$\int\limits_{\Gamma_\alpha} \Phi^2\, ds \leqslant \varkappa^2 \int\limits_{\Gamma_\alpha} \Phi_{,n}^2\, ds = \varkappa^2 \int\limits_{\Gamma_\alpha} U_{,n}^2\, ds = \varkappa^2\omega^4 \int\limits_{\Gamma_\alpha} U^2\, ds = \varkappa^2\omega^4 \int\limits_{\Gamma_\alpha} [\Phi + U(Q)]^2\, ds$$

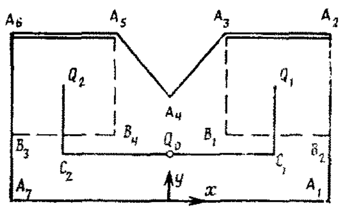
Besides, from (1.1) we find the following relations:

$$\int\limits_{\Gamma} U_{,n}\, ds = 0, \quad \int\limits_{\Gamma_\alpha} U\, ds = 0, \quad \int\limits_{\Gamma_\alpha} \Phi\, ds + U(Q) \int\limits_{\Gamma_\alpha} ds = 0$$

$$\int\limits_{\Gamma_\alpha} [\Phi + U(Q)]^2\, ds = \int\limits_{\Gamma_\alpha} \Phi^2\, ds - U^2(Q) \Big(\int\limits_{\Gamma_\alpha} ds\Big)^2 \leqslant \int\limits_{\Gamma_\alpha} \Phi^2\, ds$$

By substituting this inequality into (4.1), we obtain the result of Theorem 3.

5. As an example, a domain D is shown in the figure. D is bounded by a piecewise-smooth curve $A_1A_2\ldots A_7$ with two sections A_2A_3 and A_5A_6 representing the free surface of the liquid, which are marked by a double line. D is symmetrical about the axis $x=0$. The points A_i have the following coordinates: $A_1\,(1/2, 0)$, $A_2\,(1/2, h)$, $A_3\,(1/2 - \lambda, h)$, $A_4\,(0, \mu)$, $A_5\,(\lambda - 1/2, h)$ $A_6\,(-1/2, h)$, $A_7\,(-1/2, 0)$. It follows from geometrical considerations that $\lambda < 1/2, h > \mu$. To fix our ideas, we will set $h > \lambda$. Two star-shaped domains D_i, $i = 1,2$ are chosen in D represented by the squares $B_1B_2A_2A_3$ and $B_3B_4A_5A_6$ with centres at Q_i, where $Q_1\,(1/2 - \lambda/2,\ h - \lambda/2)$, $Q_2\,(\lambda/2 - 1/2,\ h - \lambda/2)$. The characteristic point Q_0 has the following coordinates: $Q_0\,(0, \mu/2)$.



As the curves γ_i connecting Q_0 and Q_i $(i = 1, 2)$ we choose $Q_0C_1Q_1$ and $Q_0C_2Q_2$. On the basis of (2.15), (3.2), and some geometrical considerations, we obtain the following expressions for the geometrical parameters:

$$\varkappa_1 = \varkappa_2 = 3.4\lambda, \quad l_1 = l_2 = \lambda, \quad q_1 = q_2 = \frac{1-\lambda}{\mu} + \frac{2h - \mu - \lambda}{\lambda}$$

Substituting the values of the parameters into (3.5), we can obtain the following algebraic expression for the lower estimate of the leading characteristic frequency:

$$\omega_1 \geqslant \omega_{10}, \quad \omega_{10} = \varkappa^{-1/2}$$

$$\varkappa = \lambda \left[3.4 + \frac{5.2}{\sqrt{\pi}} \left(\frac{1-\lambda}{\mu} + \frac{2h - \lambda - \mu}{\lambda}\right) + \frac{2}{\pi} \left(\frac{1-\lambda}{\mu} + \frac{2h - \lambda - \mu}{\lambda}\right)^2\right]$$

It is interesting to know how much underestimated the leading characteristic frequency is compared with the exact value. We can make this comparison in cases where the exact value of the leading characteristic frequency for the problem is known. For a channel of rectangular form with depth h and width l the exact value of the leading characteristic frequency is equal to

$$\omega_1 = \sqrt{\frac{\pi}{l} \operatorname{th} \frac{\pi}{l} h}$$

If the ratio of the depth and the width is $h/l = 1$ or $h/l = 0.25$ then the leading characteristic frequency is equal to

$$\omega_1 = 1.77l^{-1/2}, \quad \omega_1 = 1.44l^{-1/2}$$

respectively.

For $h/l = 1$, a square with side l can be chosen as the auxiliary star-shaped domain and, on the basis of (2.15), we have $\varkappa = 3.4l$. If $h/l = 0.25$, then four squares with sides $l/4$ can be chosen as the auxiliary domains, Q_0 can be chosen at the centre of the rectangle, the points Q_i can be chosen at the centres of the squares, and rectilinear intervals connecting Q_0 with Q_i can be chosen as the curves γ_i. Then, $\varkappa = 5.44l$. Therefore, the lower estimates for the characteristic frequencies for the ratios $h/l = 1$ and $h/l = 0.25$ are

$$\omega_{10} = 0.54l^{-1/2}, \quad \omega_{10} = 0.43l^{-1/2}$$

respectively.

A comparison with the exact values indicates that the characteristic frequency is underestimated approximately by a factor of three. For a cylindrical channel of diameter l and depth $h = l/2$, the approximate numerical value of the leading characteristic frequency is $\omega_1 \approx 1.4l^{-1/2}$ /2/. The lower estimate can be obtained by choosing two sectors of radius $l/2$ and central angle $\pi/2$ as the auxiliary domains. Then, by (2.15) and (3.5), $\varkappa = 6.57l$ and the lower estimate for the frequency is $\omega_{10} = 0.4l^{-1/2}$.

The value of $\varkappa$ can be reduced, and so a better lower estimate for the frequency can be obtained, by a more optimal choice of the decomposition of the given domain into a system of star-shaped domains.

In practice, it may be best to use the estimates if the domain under consideration has a complex shape and the value of the leading characteristic frequency plays the role of a restriction (not necessarily the main one) for planning the construction. In this case the optimal version of the construction can be chosen by introducing a bound for the lower estimate rather than for the frequency itself. This makes optimal computations significantly simpler, since a number of algebraic formulae can be obtained for the estimate and the difference between the frequency and its lower estimate may provide a margin for the stability of the construction. The lower estimate can also be used to carry out some preliminary approximate computations and to test computer programs for finding numerical solutions of the problems concerning oscillations of a liquid, which should produce a value of the leading characteristic frequency that exceeds the estimate obtained in Theorem 3.

REFERENCES

1. MOISEYEV N.N. and RUMYANTSEV V.V., the Dynamics of a Body with Cavities Containing Some Liquid, Nauka, Moscow, 1965.
2. MOISEYEV N.N. and PETROV A.A., Numerical methods for computing the characteristic frequencies of the oscillations of a bounded volume of a liquid, VTs, Akad. Nauk SSSR, Moscow, 1966.
3. LUKOVSKII I.A., BARNYAK M.YA. and KOMARENKO A.N., Approximate Methods for Solving Problems in the Dynamics of a Bounded Volume of a Liquid, Nauk. Dumka, Kiev, 1984.
4. HURWITZ A. and COURANT P., The Theory of Functions, Nauka, Moscow, 1968.
5. TARAKANOV V.I., On the values of constants in inequalities leading to a priori estimates in L_2 for the solution of the Neumann boundary-value problem for the Laplace equation, Izv. Vuz. Matematika, 1986.

Translated by T.Z.